The Fuzzy Sphere ⋆-Product and Spin Networks

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Abstract

We analyze the expansion of the fuzzy sphere non-commutative product in powers of the non-commutativity parameter. To analyze this expansion we develop a graphical technique that uses spin networks. This technique is potentially interesting in its own right as introducing spin networks of Penrose into non-commutative geometry. Our analysis leads to a clarification of the link between the fuzzy sphere non-commutative product and the usual deformation quantization of the sphere in terms of the ⋆-product.

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1 Introduction

This paper originated from an observation that manipulating with functions on the fuzzy sphere is equivalent to manipulating with certain SU(2) spin networks. Although this observation is nothing more than a reinterpretation of the construction [1] of non-commutative sphere spherical harmonics, it does bring spin networks of Penrose [2] into the subject of non-commutative geometry, and is thus interesting as providing an unusual perspective on non-commutative manifolds.

In this paper we would like to illustrate the usage of spin networks by deriving some facts about the non-commutative product on the fuzzy sphere. The fuzzy sphere of [3] gives one of the simplest examples of non-commutative spaces. Its structure is, in a sense, even simpler than that of the usual non-commutative plane, for the algebra of functions on the fuzzy sphere is *finite dimensional*, unlike in the case of the plane. On the other hand, as we shall see, the structure of the $\star$-product in this case is much more complicated.

In [3] the fuzzy sphere is constructed replacing the algebra of polynomials on the sphere by the non-commutative algebra generated by Pauli matrices taken in a fixed irreducible representation of SU(2). More precisely, the algebra of functions from $L^2(S^2)$ is thought of as the algebra of polynomials in $x_i \in \mathbb{R}^3$ modulo the relation $\|x\|^2 = 1$. We set the radius of the sphere to be 1. One then quantizes the coordinates x_i via:

$$x_i \to \hat{X}^i := \frac{\hat{J}^i}{\sqrt{\frac{N}{2}(\frac{N}{2}+1)}}, \tag{1.1}$$

where $\hat{J}^i$ are the generators of $\mathfrak{su}(2)$, satisfying $[\hat{J}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{J}^k$, taken in the $(N+1)$-dimensional irreducible representation of SU(2). The factor in (1.1) is adjusted precisely in such a way that the "quantized" coordinates $\hat{X}^i$ square to one. The rule (1.1) gives quantization of the monomials of order one in coordinates. Monomials of order up to N are quantized by replacing the products of the coordinates x_i by the symmetrized products of matrices $\hat{X}^i$ (see more on this map in Section 3). Monomials of order $(N+1)$ and higher become linearly dependent of lower order monomials, so that the algebra of functions on the fuzzy sphere is finite-dimensional. The integral over S^2 is replaced by the trace:

$$\int_{S^2} \to \frac{1}{(N+1)} \operatorname{Tr}. \tag{1.2}$$

When $1/N$, which plays the role of the parameter of non-commutativity, is taken to zero, the commutator of $\hat{X}^i$ vanishes, and the algebra of functions on the non-commutative sphere reduces to the commutative algebra. The trace (1.2) then reduces to the usual integral. It is in this sense that the fuzzy sphere reduces to the usual S^2 when the non-commutativity parameter is sent to zero. For more details on this construction see, e.g, [3].

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In practice one would like to have a more explicit description of the above quantization map. In particular, any function on the sphere can be decomposed into the basis of spherical harmonics, and one would like to know the matrices into which the spherical harmonics are sent under the quantization map. This was described in [1], where it was realized that the components of these matrices are given by certain Clebsch-Gordan coefficients. Interestingly, the work [1] appeared before the introduction of the notion of fuzzy sphere in [3], and contained essentially all the ingredients of the construction.

Let us now explain why one should expect the appearance of Clebsch-Gordan coefficients, or $3j$-symbols, as the result of the quantization map. Consider the following simple chain of isomorphisms:

$$\mathrm{End}(V^{\frac{N}{2}}) \sim V^{\frac{N}{2}} \times (V^{\frac{N}{2}})^* \sim \oplus_{l=1}^{N} V^l . \tag{1.3}$$

Here V^l is the space of the irreducible representation of SU(2) of the dimension $(2l+1)$. The Clebsch-Gordan coefficients relate a basis on the right hand side of (1.3) to a basis on the left hand side. A particular basis in V^l is given by the usual spherical harmonics $Y_{lm}(\theta, \phi)$. Thus, the isomorphism in (1.3) implies that every $Y_{lm}, l \leq N$ must be representable as an element of $\mathrm{End}(V^{\frac{N}{2}})$, that is, as an $(N+1) \times (N+1)$ matrix. The components of this matrix are given by the corresponding Clebsch-Gordan coefficients. We shall write down the corresponding formulas in the next section.

The appearance of the Clebsch-Gordan coefficients, or $3j$-symbols, as components of the matrices representing the spherical harmonics indicates that spin networks must play some role. Indeed, spin networks are exactly the quantities constructed from $3j$-symbols, corresponding to their vertices, with their indices contracted in some way, the contraction being represented by the spin network edges. As we shall see, the problem of calculation of the non-commutative analog of the integral of a product of a number of spherical harmonics always reduces to the evaluation of a particular spin network. One of the goals of the present paper is to develop the corresponding techniques. We do this by studying in some detail the $\star$-product on the fuzzy sphere.

Our paper partially overlaps with work [4]. In particular, the formula for the $\star$-product on the fuzzy sphere in terms of the $6j$-symbol is also contained in [4]. What is new in this paper is the expression for the expansion of this $\star$-product in powers of the non-commutativity parameter. Our approach also allows us to clarify the link between the fuzzy sphere product and the deformation quantization $\star$-product.

2 Setup

Under the quantization map the spherical harmonics $Y_{lm}(x), x \in S^2$ are mapped into certain $(N + 1) \times (N + 1)$ matrices, and, as we explained in the introduction, the components of these matrices are given by the Clebsch-Gordan coefficients. Before we spell out what these matrices are, let us fix our conventions as to what basis of spherical harmonics is used. Let us introduce, for integer l,

$$\bar{\Theta}_m^l(x) := i^{-m} \langle l, m | T_g | \omega \rangle, \tag{2.1}$$

where bar denotes the complex conjugation, $|l, m\rangle$ form a highest weight normalized basis in the irreducible representation of the dimension $(2l + 1)$, and $|\omega\rangle$ is the vector (unique up to a phase) that is invariant under the action of some fixed SO(2) subgroup of SO(3). It is given by $|\omega\rangle = |l, 0\rangle$. Then (2.1) is a function on the homogeneous space SO(3)/SO(2) $\sim S^2$. Using the formula (A.1) for the integral of the product of two matrix elements, one gets the orthogonality relation for Θ:

$$\int_{S^2} dx\, \bar{\Theta}_m^l \Theta_{m'}^{l'} = \frac{\delta^{ll'} \delta_{mm'}}{\dim_l}, \tag{2.2}$$

where dx is the normalized measure on the sphere. The presence of the factor of $\dim_l = (2l + 1)$ in this formula (and also the usage of the normalized measure dx) is what makes our Θ_m^l different from the usual spherical harmonics Y_{lm}. The basis (2.1) satisfies:

$$\bar{\Theta}_m^l = (-1)^m \Theta_{-m}^l. \tag{2.3}$$

Note that the same relation is satisfied by the usual Y_{lm}, so our basis is indeed only different by a normalization.

Along with the orthogonality relation (2.2), we will need the value of the integral of the product of three spherical harmonics. It is easily computed using the formula (A.2) for the integral of the product of three matrix elements and the definition (2.1) of the spherical harmonics. We have:

$$\int_{S^2} dx\, \overline{\Theta_{m_3}^{l_3}(x)} \Theta_{m_1}^{l_1}(x) \Theta_{m_2}^{l_2}(x) = \overline{\hat{C}_{0\,0\,0}^{l_3 l_1 l_2}} \hat{C}_{m_3 m_1 m_2}^{l_3 l_1 l_2}. \tag{2.4}$$

Here $\hat{C}$ are Clebsch-Gordan coefficients, and we have used the fact that the right hand side is only non-zero when $m_1 + m_2 = m_3$. Our choice of the normalization for Clebsch-Gordan's is such that the so-called theta-symbol is always equal to one, see (2.12). For reference, let us mention that our coefficients $\hat{C}^l$ are given by $1/\sqrt{\dim_l}$ times the Clebsch-Gordan coefficients used by Vilenkin and Klimyk [5]. The "hat" over the symbol of the coefficient

is used precisely to indicate this difference in normalizations. It is now not hard to see that (2.4), together with (2.2), implies that:

$$\Theta^{l_1}_{m_1}(x)\Theta^{l_2}_{m_2}(x) = \sum_{l_3=|l_1-l_2|}^{l_1+l_2} \sum_{m_3} \dim_{l_3} \hat{\Theta}^{l_3}_{m_3}(x) \overline{\hat{C}^{l_3 l_1 l_2}_{000}} \hat{C}^{l_3 l_1 l_2}_{m_3 m_1 m_2}. \tag{2.5}$$

Note that, in this formula, the commutativity of the product comes from the symmetry relations for the Clebsch-Gordan, namely:

$$\hat{C}^{l_3 l_2 l_1}_{m_3 m_2 m_1} = (-1)^{l_3-l_1-l_2} \hat{C}^{l_3 l_1 l_2}_{m_3 m_1 m_2}. \tag{2.6}$$

The group SO(3) acts on functions on S^2 by left shifts $T_g f(x) = f(g^{-1}x)$, and the functions $\Theta^l_m(x)$ for fixed l span the vector space V^l, in which the representation by shifts is irreducible. In view of the isomorphisms (1.3), functions $\Theta^l_m(x), l \leq N$ can be mapped to $(N+1)\times(N+1)$ matrices, whose components must be given by Clebsch-Gordan coefficients. These matrices are:

$$[\hat{\Theta}^l_m]_{ij} = \sqrt{N+1}\, \hat{C}^{l\frac{N}{2}\frac{N}{2}*}_{mij}, \tag{2.7}$$

where $\hat{C}^{l\frac{N}{2}\frac{N}{2}*}_{mij}$ are Clebsch-Gordan coefficients with properties:

$$\sum_{mij} \overline{\hat{C}^{l\frac{N}{2}\frac{N}{2}*}_{mij}} \hat{C}^{l\frac{N}{2}\frac{N}{2}*}_{mij} = 1 \tag{2.8}$$

$$\overline{\hat{C}^{l\frac{N}{2}\frac{N}{2}*}_{mji}} = (-1)^m \hat{C}^{l\frac{N}{2}\frac{N}{2}*}_{-mij}. \tag{2.9}$$

Here $(N/2)^*$ is the conjugate representation to $N/2$. The first of these properties is the normalization condition that will be explained below, while the second is the "quantum" analog:

$$(\hat{\Theta}^l_m)^\dagger = \widehat{\overline{\Theta^l_m}} \tag{2.10}$$

of the classical property (2.3).

We are now in the position to introduce the graphic notations that will lead us to spin networks and somewhat explain the normalization choices made above. We shall denote the Clebsch-Gordan's by a tri-valent vertex, with its three edges representing the three pairs of indices of $\hat{C}$, so that

$$[\hat{\Theta}^l_m]_{ij} = \sqrt{N+1} \tag{2.11}$$

Each edge corresponds to a pair of indices: an irreducible representation (spin) and a basis vector in this representation. If no spin is indicated, as for the bottom edges in (2.11), then $N/2$ is assumed. Edges are oriented. The operation of complex conjugation is represented by

changing the orientation of all the edges. The origin of the graphic notation (2.11) is evident: the Clebsch-Gordan coefficients it represents are just the matrix elements of the intertwiner between the tensor product of $V^{\frac{N}{2}} \times (V^{\frac{N}{2}})^*$ and the representation V^l; this intertwiner is represented by a trivalent vertex.

Using the graphical notation introduced, the normalization condition (2.8) is the statement about the value of the so-called theta graph:

$$\text{(theta graph with label } l\text{)} = 1. \tag{2.12}$$

This graph is constructed taking the product of two $3j$-symbols and summing over the "internal" indices, as in (2.8). The theta graph is the simplest spin network, and the $3j$ symbols we use are normalized in precisely such a way that the value of this graph is always one.

As we mentioned in the Introduction, for the non-commutative sphere, the integral over S^2 goes into the trace. Let us now find the quantum analog of the orthogonality relation (2.2). The corresponding quantity is graphically represented as:

$$\frac{1}{N+1}\text{Tr}\left(\hat{\Theta}^l_m\right)^\dagger \hat{\Theta}^{l'}_{m'} = \text{(circle diagram with labels } l'\,m' \text{ top, } l\,m \text{ bottom)} \tag{2.13}$$

Here $\dagger$ denotes the operation of taking the Hermitian conjugation, and its effect is exactly such that all "internal" indices are contracted as in the above diagram. Now using the elementary fact that:

$$\text{(circle diagram)} = \frac{1}{\dim_l} \text{(straight line with } l'\,m' \text{ and } l\,m\text{)} \text{(theta graph with label } l\text{)} \tag{2.14}$$

where the straight line represents the matrix element of the intertwiner between representations l and l', that is, the product of Kronecker deltas, we see that the "quantum" spherical harmonics $\hat{\Theta}$ satisfy exactly the same orthogonality relation (2.2) as the classical ones. The factor of $\sqrt{N+1}$ in (2.7) was adjusted precisely in such a way that this property holds.

We hope that the reader has already started appreciating the convenience of the graphical notations. Using these notations, complicated expressions are calculated by using elementary facts from representation theory. Thus, the property (2.14) is proved by noticing that the left hand side of this equation is an intertwiner between representations l and l'. Such an intertwiner only exists when $l = l'$, which means that the result must be proportional to the

straight line. The proportionality coefficient can be calculated by taking the trace of the whole expression, which is graphically represented by "closing" the open ends of the diagram. Performing this operation on the left hand side, one gets the theta graph. The right hand side gives the loop, whose value is just the dimension of the corresponding representation. We shall see other examples of such proofs in the sequel.

Let us now, before we go to our discussion of the $\star$-product on the fuzzy sphere, prove that the quantization rule (2.7) does give the correct quantization of the sphere, that is, the one given by (1.1). To this end, we must calculate the commutator of the non-commuting coordinates $\hat{X}^i$. Recall that classically the coordinates x_i are just the spherical harmonics corresponding to $l = 1$:

$$x_1 = \frac{1}{\sqrt{2}}(\Theta_1^1 - \Theta_{-1}^1), \qquad x_2 = \frac{1}{i\sqrt{2}}(\Theta_1^1 + \Theta_{-1}^1), \qquad x_3 = \Theta_0^1. \qquad (2.15)$$

In the quantum case we replace the harmonics Θ^1 by the corresponding matrices (2.7). We then must have:

$$[\hat{X}^i, \hat{X}^j] = i\epsilon^{ijk}\hat{X}^k \frac{1}{\sqrt{\frac{N}{2}(\frac{N}{2}+1)}}. \qquad (2.16)$$

Let us show that this property indeed holds. We have:

$$\frac{1}{N+1}[\hat{X}^i, \hat{X}^j] = \qquad (2.17)$$

where the spin 1 is assumed on the vertical lines, and

$$\frac{1}{(N+1)^{3/2}}\mathrm{Tr}\left((\hat{X}^k)^\dagger[\hat{X}^i, \hat{X}^j]\right) = \qquad (2.18)$$

To get the last equality we used the fact that each of the two terms is an intertwiner from a single representation of spin 1 to the tensor product of two such representations. Such an intertwiner must be proportional to the unique intertwiner that is given by the tri-valent vertex on the right hand side of (2.18). The proportionality coefficient is easily determined by closing all the open edges, and is given by the $6j$-symbol on the right hand side. The second term equals minus the first one see (2.5), which explains the factor of 2. The $3j$-symbol is given by:

$$\qquad = \frac{(-i)}{\sqrt{3!}}\epsilon^{ijk}, \qquad (2.19)$$

and the $6j$-symbol can be calculated using the formula (A.3) given in the Appendix. The result is:

$$\qquad = (-1)\frac{1}{\sqrt{3!}}\frac{1}{\sqrt{N+1}}\frac{1}{\sqrt{\frac{N}{2}(\frac{N}{2}+1)}}. \qquad (2.20)$$

Combining all this together, we see that (2.16) indeed holds.

We thus learn that the quantization rule (2.7) coincides with the standard quantization map (1.1), at least for the spherical harmonics of the first order. Interestingly, the result of the commutator in the quantum case turns out to be expressed through the $6j$-symbol. In the case considered when all the spins were taken to be $l = 1$, the commutator coincides with the classical result. However, for higher modes l one expects a deviation from the classical expressions. This is summarized in the notion of the $\star$-product. We shall now illustrate the described spin network techniques by deriving some facts about this $\star$-product.

3 The $\star_N$-product

As we have said in the introduction, our aim is to illustrate the spin network construction by deriving some facts about the fuzzy sphere non-commutative product. However, before we introduce and study this product, let us review the usual $\star$-product that arises in the deformation quantization of $\mathbb{R}^3$ equipped with Poisson structure invariant under the rotation group. We will then discuss a relation between this, and the fuzzy sphere product.

Let us start by reminding some general facts about $\star$-products. A $\star$-product gives a non-commutative deformation of the usual (point-wise) multiplication of functions. Let $\mathcal{A} = \mathcal{C}^\infty(M)$ be the space of smooth function on a manifold M, and suppose that M is equipped with a Poisson bracket $\{\cdot, \cdot\}$. Let us denote by $A[[\hbar]]$ the space of formal power series with coefficients in $\mathcal{A}$. A star product on $\mathcal{A}$ is an associative, $\mathbb{R}[[\hbar]]$-linear product on $A[[\hbar]]$. The product of two functions on M is given by

$$\phi \star \psi = \sum_n \hbar^n B_n(f, g), \tag{3.1}$$

$B_n(\cdot, \cdot)$ being bidifferential operators. The first term in the expansion is the usual commutative product $B_0(f, g) = fg$, while the first term in the commutator is the Poisson bracket $B_1(f, g) - B_1(g, f) = \{f, g\}$. The Poisson bracket therefore gives the germ of deformation of the commutative product towards $\star$-product. There is a notion of gauge transformation that can be defined on the set of $\star$-products and Poisson brackets. The group of these gauge transformations consists of linear automorphisms of $\mathcal{A}[[\hbar]]$ of the following form:

$$f \to U(f) = f + \hbar U_1(f) + \cdots + \hbar^n U_n(f), \tag{3.2}$$

where $U_i(f)$ are differential operators. It acts on the set of star product as

$$\phi \star_U \psi = U^{-1}(U(\phi) \star U(\psi)). \tag{3.3}$$

Products related by a such transformation are called equivalent.

As is well known, if M is the dual of a Lie algebra, $M = \mathfrak{g}^*$ there is the so-called Kirillov-Lie Poisson structure on it, whose symplectic leaves are coadjoint orbits. The bracket is

$$\{\phi, \psi\}(x) = x^k C_k^{ij} \partial_i \phi(x) \partial_j \psi(x), \tag{3.4}$$

where x^i are the components of $x \in \mathfrak{g}^*$. Using the natural identification between linear functions on $\mathfrak{g}^*$ and $\mathfrak{g}$ one can equivalently view x^i as a basis e^i in $\mathfrak{g}$. Then C_k^{ij} are the structure constants with respect to this basis. This identification extends to a natural isomorphism between the space $\mathrm{Pol}(\mathfrak{g}^*)$ of polynomials in x^i and symmetric algebra $\mathrm{Sym}(\mathfrak{g}^*)$. There are several equivalent ways to introduce a deformation quantization of the Kirillov bracket. We shall refer to the arising $\star$-product as Kirillov $\star$-product.

Universal enveloping algebra

The standard way to introduce a $\star$-product in $\mathfrak{g}^*$ is by using the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Recall that the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with generators e^i satisfying

$$[e^i, e^j] = \hbar C_k^{ij} e^k, \tag{3.5}$$

where C_k^{ij} are the structure constants, is the algebra generated by all polynomials in the generators e^i modulo the relation $XY - YX = [X, Y]$ for all $X, Y \in \mathfrak{g}$. The Poincare-Birkhoff-Witt theorem states that $\mathfrak{U}$ is isomorphic to the algebra $\mathrm{Sym}(\mathfrak{g}) \sim \mathrm{Pol}(\mathfrak{g}^*)$ generated by all completely symmetrized polynomials in the generators: $\mathfrak{U}(\mathfrak{g}) \sim \mathrm{Sym}(\mathfrak{g})$. Since the algebra $\mathrm{Sym}(\mathfrak{g})$ is naturally isomorphic to the algebra $\mathrm{Pol}(\mathfrak{g}^*)$ of polynomial functions on the dual space $\mathfrak{g}^*$, we have the one-to-one map

$$\sigma : \mathrm{Pol}(\mathfrak{g}^*) \to \mathfrak{U}(\mathfrak{g}),$$
$$x^{k_1} \ldots x^{k_n} \to \frac{1}{n!} \sum_s e^{k_{s(1)}} \ldots e^{k_{s(n)}},$$

where the sum is taken over all permutations s of n integers, x^i are linear functionals on $\mathfrak{g}^*$, and e^i are the corresponding elements in $\mathfrak{g}$. Using the map σ, one can define the following non-commutative product on $\mathrm{Pol}(\mathfrak{g}^*)$:

$$(\phi \star \psi)(x) = \sigma^{-1} \left[\sigma(\phi) \sigma(\psi) \right]. \tag{3.6}$$

Note that this $\star$-product explicitly depends on the deformation parameter $\hbar$ because the later enters the commutator (3.5). It is not so straightforward to see that this product can be expanded in terms of differential operators. For example, one can prove the following formula:

$$(e^X \star Y)(x) = e^{X(x)} \left[\frac{\mathrm{ad}X}{1 - e^{-\mathrm{ad}X}} \cdot Y \right](x). \tag{3.7}$$

Baker-Campbell-Haussdorf formula

The Baker-Campbell-Haussdorf formula states that, given two Lie algebra elements X, Y, the product of the corresponding group elements can be expressed as:

$$e^X \cdot e^Y = e^{X+Y+\Psi(X,Y)}, \tag{3.8}$$

where $X, Y, \Psi(X, Y) \in \mathfrak{g}$ and

$$X + Y + \Psi(X,Y) = \sum_{m=1}^{\infty} \frac{(-1)^{(m-1)}}{m} \sum_{\substack{k_i + l_i \geq 1 \\ k_i \geq 0, l_i \geq 0}} \frac{[X^{k_1} Y^{l_1} \cdots X^{k_m} Y^{l_m}]}{k_1! l_1! \cdots k_m! l_m!}, \tag{3.9}$$

and $[X_1 \cdots X_n]$, $X_i \in \mathfrak{g}$ denotes $\frac{1}{n}[\cdots [[X_1, X_2], X_3], \cdots, X_n]$. This is the so-called Campbell-Hausdorff formula in the Dynkin form, see, e.g., [6]. Representing the commutator as $[X, Y] = XY - YX$, one can think of $\Psi(X, Y)$ as a complicated sum of polynomials in the components X_i, Y_i:

$$\Psi_i(X, Y) = \sum_{i_{\vec{n}} j_{\vec{m}}} A_i^{i_{\vec{n}} j_{\vec{m}}} X_{i_1} \cdots X_{i_n} Y_{j_1} \cdots Y_{j_m}. \tag{3.10}$$

One then defines a $\star$-product on the space of functions on $\mathbb{R}^3 = \mathfrak{g}^*$ by:

$$(\phi \star \psi)(x) = \phi \, e^{\frac{1}{\hbar}(x|\Psi(\hbar\partial, \hbar\partial))} \psi, \tag{3.11}$$

where we have replaced the argument X_i of $\Psi(X, Y)$ by $\hbar$ times the derivative $\partial/\partial x^i$ acting on the left on ϕ, and the argument Y_i by the derivative acting on the right on ψ. The quantity $(x|\Psi)$ in the exponential stands for the contraction $x^i \Psi_i$. One can prove that the product (3.11) is, in fact, the same as (3.6). A good exposition of the relation between these two quantizations, and also of the relation of both to the Kontsevich deformation quantization, is given in [7].

Path integral

The third way to define Kirillov $\star$-product uses a version of Cattaneo-Felder [8] path integral. For the case in question, the function $\alpha^{ij}(x)$ that determines the Poisson structure:

$$\{\phi, \psi\} = \sum_{ij} \alpha^{ij}(x) \partial_i \phi(x) \partial_j \psi(x) \tag{3.12}$$

is linear in x, and the action used in [8] can be given a simple form of the action of the so-called BF theory. Thus, let us consider the following theory on the unit disc. It has two dynamical fields: the field B_i, which is $\mathfrak{g}^*$ valued, this field is the analog of the field X^i of

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[8], and the G-connection A^i, with curvature $F^i(A)$. This connection is the analog of the one-form field η of [8]. The action is then given by:

$$S[B, A] = \int_U B_i F^i(A),\tag{3.13}$$

where the integral is taken over the unit disc U. The $\star$-product is then given by the following path integral:

$$(\phi \star \psi)(x) = \langle \phi(B(0))\psi(B(1))\rangle_x \equiv \int_{B(\infty)=x} \mathcal{D}B\mathcal{D}A\, \phi(B(0))\psi(B(1))\, e^{\frac{1}{\hbar}\int BF}.\tag{3.14}$$

Thus, the $\star$-product is obtained by computing the correlation function of two operators in this theory. The operators are given by functions ϕ, ψ on the values of the B-field at two boundary points $0, 1 \in \partial U$. The value of B at the third boundary point ∞ is kept fixed in the path integral. One also has the boundary conditions for the connection: it is required that the connection one-form vanishes on ∂U on vectors tangent to the boundary. Then the perturbative expansion of the path integral (3.14) gives a non-commutative product in $\mathbb{R}^3$, which can be checked to be associative, and, thus, is a $\star$-product. Since the BF action (3.13) is essentially the one used by Cattaneo and Felder [8], the product (3.14) is the Kontsevich product [9]. It can then be shown, see [7], that this product is equivalent to the one defined using the Campbell-Hausdorff formula.

$\star_N$-Product

Having reviewed the usual $\star$-product, let us introduce the non-commutative product that is relevant in the context of the fuzzy sphere. The square integrable functions on S^2 can be decomposed into the basis of spherical harmonics:

$$\phi(x) = \sum_{lm} \phi_m^l \Theta_m^l(x).\tag{3.15}$$

We have a quantization map, which sends spherical harmonics $\Theta_m^l, l \leq N$ to matrices (2.11). Under this map, functions are sent to matrices:

$$\phi(x) \to \hat{\phi} = \sum_{l=0}^N \sum_m \phi_m^l \hat{\Theta}_m^l\tag{3.16}$$

Note that this map is insensitive to the "high frequency" behavior of the function, for it cuts off all the harmonics with $l > N$.

Let us also construct the inverse map. To this end, we introduce the non-commutative analog of δ-function:

$$\hat{\delta}_x = \sum_{l=0}^N \sum_m \dim_l (\hat{\Theta}_m^l)^\dagger \Theta_m^l(x).\tag{3.17}$$

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Thus, the "quantum" δ-function is a matrix that in addition depends on a point $x \in S^2$. Note that the δ-function is "real": $\hat{\delta}^\dagger = \hat{\delta}$. Given an arbitrary operator (matrix) $\hat{\phi}$ one can construct from it a square integrable function:

$$\hat{\phi} \to \phi(x) = \frac{1}{N+1} \mathrm{Tr}\left(\hat{\delta}_x \hat{\phi}\right). \tag{3.18}$$

The resulting functions, of course, contain only the modes with $l \le N$. Using the δ-function (3.17) one can also give another expression for the quantization rule (3.16). Indeed, we have:

$$\hat{\phi} = \int_{S^2} dx\, \hat{\delta}_x \phi(x). \tag{3.19}$$

One can easily check that the composition of the quantization map (3.16) and its inverse (3.18) give back the function one started from with all its modes $l > N$ cut off. Let us formalize this introducing the notion of the projector $\mathcal{P}_N$:

$$\phi \to \hat{\phi} \to \frac{1}{N+1} \mathrm{Tr}\left(\hat{\delta}_x \hat{\phi}\right) = \mathcal{P}_N \phi. \tag{3.20}$$

Thus, the maps (3.16) and (3.18) are one-to-one on the space of functions that only contain modes up to N. Let us denote this space by $\mathcal{A}^N$. Note that $\mathcal{A}^N = \mathcal{A}^\infty / \mathrm{Ker}\, \mathcal{P}_N$, where $\mathcal{A}^\infty$ is the algebra of all L^2 functions on S^2.

Having defined the quantization map, we can use it to define a non-commutative product on the space $\mathcal{A}^N$ via:

$$\widehat{\phi \star_N \psi} = \hat{\phi}\hat{\psi}. \tag{3.21}$$

Let us emphasize that this is well defined only in $\mathcal{A}^N$ since $\hat{\phi} = \widehat{\mathcal{P}_N \phi}$ and the product $\star_N$ does not coincide with the usual $\star$-product reviewed above, because they are defined on different spaces. While $\star_N$ is the product on the space of functions $\mathcal{A}^N$, the "real" $\star$-product acts on the space of all square integrable functions $\mathcal{A}^\infty$. However, we are going to show that the $\star_N$-product is related to the usual $\star$-product via:

$$\phi \star_N \psi = \mathcal{P}_N \left[\phi \star_\hbar \psi\right]\Big|_{\hbar = 2/(N+1)}. \tag{3.22}$$

Let us note that the interplay between the non-commutative product and deformation quantization was studied before, see, e.g., [10]. What is new in this paper is the justification for the formula (3.22) coming from an explicit expression for the asymptotic expansion of the $\star_N$-product in powers of the non-commutativity parameter. To find this expansion we consider the $\star_N$-product of the modes. The product of two matrices $\hat{\Theta}$ can be decomposed into a sum of $\hat{\Theta}$. A simple calculation, similar to the one performed in (2.17), (2.18), gives:

$$\Theta^{l_1}_{m_1} \star_N \Theta^{l_2}_{m_2} = \sum_{l_3=0}^{N} \sum_{m_3} \mathrm{dim}_{l_3} \Theta^{l_3}_{m_3} \quad \raisebox{-1em}{\includegraphics{}} \quad \sqrt{N+1} \quad \raisebox{-1em}{\includegraphics{}} \tag{3.23}$$

Let us notice that the right hand side depends on N only through the $6j$-symbol (and the square root), and the cutoff in the sum. Therefore, the non-trivial information about the $1/N$ expansion of the $\star_N$-product is all contained in the $6j$-symbol. Thus, the fuzzy sphere gives us an interesting example of a non-commutative manifold for which the $\star$-product is not only known in terms of its $1/N$ derivative expansion, but also in closed form, in terms of the $6j$-symbol in (3.23).

Let us note that (3.23) can also be written as:

$$\Theta_{m_1}^{l_1} \star_N \Theta_{m_2}^{l_2} = \sum_{l_3=0}^{N} \mathcal{P}^{(l_3)} \left(\Theta_{m_1}^{l_1} \cdot \Theta_{m_2}^{l_2} \right) \Psi_{(N)}(l_1, l_2, l_3), \tag{3.24}$$

where $\mathcal{P}^{(l)}$ is the projector on l-th representation

$$\mathcal{P}_N = \sum_{l \leq N} \mathcal{P}^{(l)}, \tag{3.25}$$

and

$$\Psi_{(N)}(l_1, l_2, l_3) := \frac{\sqrt{N+1}}{\hat{C}_{000}^{l_3 l_1 l_2}} \;\; \tag{3.26}$$

As we show below,

$$\lim_{N \to \infty} \sqrt{N+1} \;\; = \hat{C}_{000}^{l_3 l_1 l_2}. \tag{3.27}$$

This implies that the expansion of the function $\Psi_{(N)}(l_1, l_2, l_3)$ in powers of $1/N$ starts from one. This fact, together with the relation (2.5) means that the zeroth order term in the $1/N$ expansion of the $\star_N$-product of two modes Θ is given by the usual product, which is what one expects.

Before we look more closely into the details of the asymptotic expansion, let us show that the non-commutative product defined by (3.23) is indeed an associative product. Interestingly, the associativity follows from the so-called Biedenharn-Elliott (or pentagon) identity, which reads:

$$\sum_l \dim_l \;\; = \;\; \tag{3.28}$$

Note that although one has to sum over all l in this formula, only the terms with $l \leq N$ survive. We are also going to use the following recoupling identity:

$$ = \sum_{l_6} \dim_{l_6} \;\; \tag{3.29}$$

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Here the sum is taken over all l_6. The proof of the associativity is then as follows:

$$
\begin{aligned}
\left(\Theta^{l_1} \star_N \Theta^{l_2}\right) \star_N \Theta^{l_3} &= \sum_{l_3'} \dim_{l_3'} \Theta^{l_3'} \star_N \Theta^{l_3} \quad = \\[2mm]
&= \sum_{l_3' \, l_4} \dim_{l_3'} \dim_{l_4} \Theta^{l_4} \quad = \\[2mm]
&= \sum_{l_3' \, l_3'' \, l_4} \dim_{l_3'} \dim_{l_3''} \dim_{l_4} \Theta^{l_4} \quad = \\[2mm]
&= \sum_{l_3'' \, l_4} \dim_{l_3''} \dim_{l_4} \Theta^{l_4} \quad = \Theta^{l_1} \star_N \left(\Theta^{l_2} \star_N \Theta^{l_3}\right).
\end{aligned}
\tag{3.30}
$$

Here we used the recoupling identity (3.29) to get the third line, and the pentagon identity (3.28) to get the last line. Note that, although in the third line the sum is taken over all l_3'', in the last line, after we used the pentagon identity, only the terms with $l_3'' \leq N$ are non-zero. Thus, the associativity of the product $\star_N$ is intimately related to the associativity in the category of irreducible representations of SU(2), of which the Biedenharn-Elliot identity is a manifestation.

Let us now study the asymptotic expansion of the $\star_N$-product more closely. To this end, let us rewrite the formula (A.3) for the $6j$-symbol in a form that is convenient for taking the large N-limit. It is convenient to introduce, for $N + a \geq 0$:*

$$
\gamma(a; N) = \frac{(N + a)!}{N!(N + 1)^a}.
\tag{3.31}
$$

Since, for $a > 0$,

$$
\gamma(a; N) = \prod_{k=1}^{a}\left(1 + \frac{k - 1}{N + 1}\right),
\tag{3.32}
$$

this function, is analytic in $1/(N + 1)$, and, for positive a, is a polynomial of degree $a - 1$. We have:

$$
\gamma(a; N) = 1 + \frac{1}{N + 1}\frac{a(a - 1)}{2} + \ldots + \left(\frac{1}{N + 1}\right)^{a-1}(a - 1)!
\tag{3.33}
$$

---

*It is not hard to recognize this function as related to the one appearing in the infinite limit Euler definition of Γ function. Indeed, $\gamma(z; n) = \Gamma(z - 1)/F(z - 1, n + 1)$, where $F(z, n) = n^z n!/(z(z + 1)\ldots(z + n)) = n^z \int_0^1 (1 - u)^n u^{z-1} du$, and $\lim_{n\to\infty} F(z, n) = \Gamma(z)$.

One can similarly rewrite the function $\gamma(-a; N), a > 0$ as one over a polynomial of degree a:

$$\gamma(a; N) = \frac{1}{\prod_{k=1}^{a} \left(1 - \frac{k}{N+1}\right)}. \tag{3.34}$$

Using this function, the $6j$-symbol can be rewritten as:

$$
\begin{aligned}
\text{\Large$\bigcirc$} &= (-1)^{l_1}(N+1)^{-1/2}l_3! \left[\frac{(l_1 + l_2 - l_3)!}{(l_1 - l_2 + l_3)!(l_2 - l_1 + l_3)!(l_1 + l_2 + l_3 + 1)!}\right]^{1/2} \\
&\quad \left[\frac{\gamma(-l_1; N)\gamma(l_1 + 1; N)\gamma(l_3 + 1; N)}{\gamma(-l_2; N)\gamma(-l_3; N)\gamma(l_2 + 1; N)}\right]^{1/2} \\
&\quad \sum_k \frac{(-1)^k}{k!} \frac{(l_1 + k)!(l_2 + l_3 - k)!}{(l_1 - k)!(l_2 - l_3 + k)!(l_3 - k)!} \frac{\gamma(-l_3 + k; N)}{\gamma(k + 1; N)}.
\end{aligned}
\tag{3.35}
$$

The sum here is restricted to those k for which the quantities inside the factorials are non-negative. Using the representations (3.32), (3.34) for the γ function, it is not hard to deduce the analyticity properties of the $6j$-symbol as a function of $1/(N+1)$. We will need these properties below, when we discuss the relation between $\star_N$ and the usual $\star$-product. Function $\gamma(a + 1; N), a > 0$, viewed as a function of $1/(N+1)$ has simple zeros on the negative axes, with the closest to origin zero located at $1/(N+1) = -1/a$. Function $\gamma(-a; N), a$ has simple poles on the positive axes, with the closest to the origin pole located at $1/(N+1) = 1/a$. Then, the expression under the square root in (3.35) can be shown to have at most simple poles and zeros on the positive axes and at most second order zeros on the negative axes. The function inside the sum in (3.35) has simple poles both on the positive and negative axes. This proves that the $6j$-symbol times $\sqrt{N+1}$ (3.35) is an analytic function of $1/(N+1)$ in the open disc of radius $\min(1/l_1, 1/l_2, 1/l_3)$ around zero. Moreover, all singularities are located on the positive and negative axes. This means that the function $\Psi_{(N)}(l_1, l_2, l_3)$ introduced in (3.26) can be analytically continued, as a function of $1/(N+1)$, to the whole complex plane. Introducing $\hbar = 2/(N+1)$ we shall denote the analytic continuation of $\Psi_{(N)}(l_1, l_2, l_3)$ by $\Psi(l_1, l_2, l_3; \hbar)$.

It is not hard to find first terms of the expansion of $6j$ in powers of $1/(N+1)$. We get:

$$
\begin{aligned}
\sqrt{N+1} \; \text{\Large$\bigcirc$} &= \hat{C}_{000}^{l_1 l_2 l_3} \left(1 + \frac{1}{(N+1)} \frac{1}{2}(C_{l_1} - C_{l_2} + C_{l_3})\right) \\
&\quad - \frac{(-1)^{l_1}}{(N+1)}(l_3 + 1)! \left[\frac{(l_1 + l_2 - l_3)!}{(l_1 - l_2 + l_3)!(l_2 - l_1 + l_3)!(l_1 + l_2 + l_3 + 1)!}\right]^{1/2} \\
&\quad \sum_k \frac{(-1)^k}{k!} k \frac{(l_1 + k)!(l_2 + l_3 - k)!}{(l_1 - k)!(l_2 - l_3 + k)!(l_3 - k)!} + O(1/N^2),
\end{aligned}
\tag{3.36}
$$

where we have introduced a notation $C_l = l(l+1)$ and we have used the expression (A.4) for $C_{000}^{l_3 l_1 l_2}$. This proves that the zeroth order term is given by the usual commutative product.

Let us introduce a notation for the prefactor in front of the sum:

$$\rho(l_3, l_1, l_2) = (-1)^{l_1} l_3! \left[\frac{(l_1 + l_2 - l_3)!}{(l_1 - l_2 + l_3)!(l_2 - l_1 + l_3)!(l_1 + l_2 + l_3 + 1)!} \right]^{1/2}. \quad (3.37)$$

Then it can be noted that the sum in (3.36) is related to a certain Clebsch-Gordan coefficient. Namely,

$$\hat{C}^{l_3 l_1 l_2}_{1\,1\,0} = \frac{1}{l_3} \left(\frac{C_3}{C_1} \right)^{1/2} \rho(l_3, l_1, l_2) \sum_k \frac{(-1)^k}{k!} \frac{(l_1 + k)!(l_2 + l_3 - k)!}{(l_1 - k)!(l_2 - l_3 + k)!} \frac{k}{(l_3 - k)!}. \quad (3.38)$$

Therefore, the first order term in the expansion, that is, the coefficient of $1/(N+1)$ term in (3.36) can be written as:

$$\frac{1}{2}(C_1 - C_2 + C_3)\hat{C}^{l_3 l_1 l_2}_{0\,0\,0} - (C_1 C_3)^{1/2} \hat{C}^{l_3 l_1 l_2}_{1\,1\,0}. \quad (3.39)$$

It is now possible to prove the following two identities on Clebsch-Gordan coefficients,

$$\begin{aligned}
(C_1 - C_2 + C_3)\hat{C}^{l_3 l_1 l_2}_{0\,0\,0} &= (C_1 C_3)^{1/2}(\hat{C}^{l_3 l_1 l_2}_{1\,1\,0} + \hat{C}^{l_3 l_1 l_2}_{-1\,-1\,0}), & (3.40) \\
\hat{C}^{l_3 l_1 l_2}_{1\,1\,0} &= (-1)^{l_1 + l_2 + l_3} \hat{C}^{l_3 l_1 l_2}_{-1\,-1\,0}.
\end{aligned}$$

The first equality is just the expression of the intertwining property of the Clebsch-Gordan coefficient, and the second is a symmetry relation.

These two identities imply that (3.39) is equal to zero if $l_1 + l_2 + l_3$ is even. It is also clear that for odd $l_1 + l_2 + l_3$ it equals to

$$- (C_1 C_3)^{1/2} \hat{C}^{l_3 l_1 l_2}_{1\,1\,0}. \quad (3.41)$$

However, as is shown in the Appendix, this quantity is just $P(l_1, l_2, l_3)$, which is the coefficient that appears in the decomposition of the Poisson bracket of two spherical harmonics. This proves that the first order term is given by the Poisson bracket.

We can summarize all of the above results in a form of the following theorem.

Theorem 1 *There exists a function $\Psi(l_1, l_2, l_3; \hbar)$, given by the analytic continuation of the function $\Psi_{(N)}(l_1, l_2, l_3)$ introduced in (3.26), which is analytic in $\hbar$ in the open disc of radius $\min(1/l_1, 1/l_2, 1/l_3)$ around zero, and*

$$\Psi(l_1, l_2, l_3; \hbar) = 1 + \hbar \Psi_1(l_1, l_2, l_3) + (\hbar)^2 \Psi_2(l_1, l_2, l_3) + \dots \quad (3.42)$$

For $N \geq l_1, l_2, l_3$,

$$\Psi(l_1, l_2, l_3; \hbar)\Big|_{\hbar = 2/(N+1)} = \Psi_{(N)}(l_1, l_2, l_3). \quad (3.43)$$

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The product defined via:

$$\Theta_{m_1}^{l_1} \star_\hbar \Theta_{m_2}^{l_2} = \sum_{l_3=0}^{N} \mathcal{P}^{(l_3)} \left(\Theta_{m_1}^{l_1} \cdot \Theta_{m_2}^{l_2} \right) \Psi(l_1, l_2, l_3; \hbar) \tag{3.44}$$

is an associative product. Moreover,

$$\phi \star_\hbar \psi - \psi \star_\hbar \phi = \hbar\{\phi, \psi\} + o(\hbar),$$
$$\mathcal{P}_N[\phi] \star_N \mathcal{P}_N[\psi] = \mathcal{P}_N \left[\phi \star_\hbar \psi \right] \Big|_{\hbar=2/(N+1)}. \tag{3.45}$$

The associativity follows from the analytic continuation of Biedenharn-Elliot identity, see (3.28). What we did not prove is that the non-commutative product (3.44) is indeed a $\star$-product, that is given by the expansion in terms of derivatives. This is, in principle, possible with our techniques by considering the higher terms in the expansion of the $6j$-symbol. Thus, modulo this caveat, the $\star_\hbar$-product gives a deformation quantization of Kirillov bracket, and must thus be equivalent to the usual Kirillov $\star$-product. The formula (3.45) then establishes a relation between the fuzzy sphere product and Kirillov product.

We conclude by observing that the expressions (3.23), (3.24) for the $\star_N$-product can be used to define a non-commutative product on the so-called q-deformed fuzzy sphere. While the usual fuzzy sphere is defined as the structure covariant under the action of the group SU(2), its q-deformed analog is covariant under the action of the quantum group $U_q(su(2))$. For a definition of the q-deformed fuzzy sphere see, e.g., [11]. The expression (3.23) can then be used to get the product on the q-deformed sphere. To this end, one should replace all objects appearing on the right hand side of (3.23) –the dimension, the $3j$ and $6j$-symbols– by the corresponding q-deformed quantities. One gets a q-deformed product. This product is still associative, for the proof (3.30) of associativity based on the pentagon identity holds for the quantum group as well. This q-deformed product was discussed in [4], where the usual fuzzy sphere version (3.23) is also mentioned. It would be quite interesting to obtain an analog of our asymptotic formula for the q-deformed product.

Acknowledgments

A   Some formulas

In the main text we use the values of the integrals of the product of two and three matrix elements. These are given by:

$$\int_G dg \,\overline{\langle l_1, m_1|T_g|l_1, m_1'\rangle} \langle l_2, m_2|T_g|l_2, m_2'\rangle = \frac{\delta^{l_1 l_2}\delta_{m_1 m_2}\delta_{m_1' m_2'}}{\dim_{l_1}}, \tag{A.1}$$

and

$$\int_G dg \,\overline{\langle l_3, m_3|T_g|l_3, m_3'\rangle} \langle l_1, m_1|T_g|l_1, m_1'\rangle \langle l_2, m_2|T_g|l_2, m_2'\rangle = \hat{C}^{l_3 l_1 l_2}_{m_3' m_1' m_2'} \overline{\hat{C}^{l_3 l_1 l_2}_{m_3 m_1 m_2}}. \tag{A.2}$$

Let us now give some explicit formulas for the $3j$ and $6j$-symbols. All these formulas are from [5], with normalizations properly adjusted to match our conventions. We have:

$$\begin{aligned}
\bigcirc &= (-1)^{l_1} l_3! \left[\frac{(l_1 + l_2 - l_3)!}{(l_1 - l_2 + l_3)!(l_2 - l_1 + l_3)!(l_1 + l_2 + l_3 + 1)!}\right]^{1/2} \\
&\quad \left[\frac{(N - l_1)!(N + l_1 + 1)!(N + l_3 + 1)!}{(N + l_2 + 1)!(N - l_2)!(N - l_3)!}\right]^{1/2} \\
&\quad \sum_k \frac{(-1)^k}{k!} \frac{(l_1 + k)!(l_2 + l_3 - k)!}{(l_1 - k)!(l_2 - l_3 + k)!(l_3 - k)!} \frac{(N - l_3 + k)!}{(N + k + 1)!}.
\end{aligned} \tag{A.3}$$

Here the sum is taken over all k such that the factorials are taken of non-negative integers. Using this formula it is not hard to get the value (2.20) of the $6j$-symbol for all spins being equal to 1. In this case the sum in (A.3) is taken only over two values $k = 0, 1$ and the calculation leading to (2.20) is straightforward.

Let us also give an expression for the $3j$-symbol $C^{l_3 l_1 l_2}_{000}$. It can be obtained from the general expression for the $3j$-symbol (in terms of a finite sum) given in [5]. Taking into account the difference in normalizations, we get:

$$\hat{C}^{l_3 l_1 l_2}_{000} = \rho(l_3, l_1, l_2) \sum_k \frac{(-1)^k}{k!} \frac{(l_1 + k)!(l_2 + l_3 - k)!}{(l_1 - k)!(l_2 - l_3 + k)!(l_3 - k)!}, \tag{A.4}$$

where

$$\rho(l_3, l_1, l_2) = (-1)^{l_1} l_3! \left[\frac{(l_1 + l_2 - l_3)!}{(l_1 - l_2 + l_3)!(l_2 - l_1 + l_3)!(l_1 + l_2 + l_3 + 1)!}\right]^{1/2}. \tag{A.5}$$

We should also note the formula:

$$\hat{C}^{l l_1 l_2}_{000} = \frac{(-1)^{g-l} g! \Delta(l_1, l_2, l)}{(g - l_1)!(g - l_2)!(g - l)!}, \tag{A.6}$$

where $l_1 + l_2 + l = 2g, g \in \mathbb{Z}$, and $\Delta(l_1, l_2, l)$ is given by

$$\Delta(l_1, l_2, l) = \left[\frac{(l_1 + l_2 - l)!(l_1 - l_2 + l)!(l_2 - l_1 + l)!}{(l_1 + l_2 + l + 1)!}\right]^{1/2}. \tag{A.7}$$

B Poisson bracket of spherical harmonics

In this section we calculate the Poisson bracket of spherical harmonics. The result we obtain here is compared in section 3 with the first order term in the expansion of the $6j$-symbol in powers of $1/N$. We were not able to find a result for this Poisson bracket in the literature, so we sketch the calculation here.

We begin with some notations. Let J_i be generators of the Lie algebra of SU(2): $[J_i, J_j] = i\epsilon_{ijk}J_k$. These generators can be realized as vector fields in $\mathbb{R}^3$. Denoting by x_i the usual Cartesian coordinates in $\mathbb{R}^3$, we get:

$$J_i = \frac{1}{i}\epsilon_i{}^{jk}x_j\partial_k. \tag{B.1}$$

To calculate the Poisson bracket, it is convenient to introduce a set of complex coordinates in $\mathbb{R}^3$. We define $z = (x_1 + ix_2)/\sqrt{2}, x = x_3$ and $J_\pm = (J_1 \pm iJ_2)/\sqrt{2}, J = J_3$. These new generators can be expressed in terms of the complex vector fields $\partial_z, \partial_{\bar{z}}, \partial_x$. We have:

$$J_+ = x\partial_{\bar{z}} - z\partial_x, \qquad J_- = \bar{z}\partial_x + x\partial_z, \qquad J = z\partial_z - \bar{z}\partial_{\bar{z}}. \tag{B.2}$$

These vector fields satisfy:

$$[J_+, J_-] = J, \qquad [J, J_\pm] = \pm J_\pm, \tag{B.3}$$

and

$$\overline{J_+} = -J_-, \qquad \overline{J} = -J. \tag{B.4}$$

The spherical harmonics Θ^l_m are given by:

$$\Theta^l_m = \alpha^l_m J_-^{l-m}v, \tag{B.5}$$

where v is the highest weight vector $v = z^l$. To calculate the normalization factor α^l_m let us consider the norm $||J_-^{l-m}v||^2$. Using

$$[J_+, J_-^n]v = \frac{n}{2}(2l - n + 1)J_-^{n-1}v, \tag{B.6}$$

we get

$$||J_-^{l-m}v||^2 = (-1)^{l-m}\langle v|J_+^{l-m}J_-^{l-m}|v\rangle = (-1)^{l-m}\frac{1}{2^{l-m}}\frac{(l-m)!}{(l+m)!}(2l)!||v||^2. \tag{B.7}$$

An explicit calculation, using the normalized measure on S^3, gives:

$$||v||^2 = ||z^l||^2 = \int \sin^{2l}\theta = \frac{1}{(2l+1)}\frac{l!}{(2l-1)!!}. \tag{B.8}$$

Combining these two facts, and taking into account the normalization of Θ_m^l (2.2), we see that α_m^l is, up to a phase, given by:

$$\frac{1}{l!}\left(\frac{(l+m)!}{2^m(l-m)!}\right)^{1/2}.\tag{B.9}$$

We choose the phase factor in such a way that Θ_m^l coincide with the ones given by (2.1). This gives:

$$\alpha_m^l=\frac{(-1)^{l-m}}{l!}\left(\frac{(l+m)!}{2^m(l-m)!}\right)^{1/2}.\tag{B.10}$$

It is now straightforward to work out the action of vector fields $\partial_z,\partial_{\bar{z}},\partial_x$ on Θ_m^l. We get:

$$\partial_z\Theta_m^l=\sqrt{\frac{1}{2}(l+m)(l+m-1)}\Theta_{m-1}^{l-1},$$

$$\partial_{\bar{z}}\Theta_m^l=-\sqrt{\frac{1}{2}(l-m)(l-m-1)}\Theta_{m+1}^{l-1},\tag{B.11}$$

$$\partial_x\Theta_m^l=\sqrt{(l+m)(l-m)}\Theta_m^{l-1}.$$

We will also need the action of SU(2) generators on Θ_m^l:

$$J_+\Theta_m^l = -\sqrt{\frac{1}{2}(l-m)(l+m+1)}\Theta_{m+1}^l,$$

$$J_-\Theta_m^l = -\sqrt{\frac{1}{2}(l+m)(l-m+1)}\Theta_{m-1}^l,\tag{B.12}$$

$$J\Theta_m^l = m\Theta_m^l.$$

The Poisson bracket is given by:

$$\{f,g\}=\frac{i}{\sin\theta}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\varphi}-\frac{\partial g}{\partial\theta}\frac{\partial f}{\partial\varphi}\right).\tag{B.13}$$

It is normalized so that $\{x_1,x_2\}=ix_3$. It can be expressed in terms of vector fields and generators as:

$$\{f,g\}=\partial_z f(J_+g)+\partial_{\bar{z}}f(J_-g)+\partial_x f(Jg).\tag{B.14}$$

Using the expressions (B.11), (B.12) it is now straightforward to compute the Poisson bracket of two spherical harmonics. We have:

$$\{\Theta_{m_1}^{l_1},\Theta_{m_2}^{l_2}\}=-\frac{1}{2}\Big(\sqrt{(l_1+m_1)(l_1+m_1-1)(l_2-m_2)(l_2+m_2+1)}\Theta_{m_1-1}^{l_1-1}\Theta_{m_2+1}^{l_2}$$

$$-\sqrt{(l_1-m_1)(l_1-m_1-1)(l_2+m_2)(l_2-m_2+1)}\Theta_{m_1+1}^{l_1-1}\Theta_{m_2-1}^{l_2}\quad\text{(B.15)}$$

$$-2\sqrt{(l_1+m_1)(l_1-m_1)}\,m_2\Theta_{m_1}^{l_1-1}\Theta_{m_2}^{l_2}\Big).$$

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Thus, using (2.4) we have:

$$\int \{\Theta_{m_1}^{l_1}, \Theta_{m_2}^{l_2}\}\overline{\Theta_m^l} = -\frac{1}{2}\overline{\hat{C}_{00\ 0}^{ll_1-1\,l_2}}$$
$$\left(\sqrt{(l_1+m_1)(l_1+m_1-1)(l_2-m_2)(l_2+m_2+1)}\hat{C}_{m\,m_1-1\,m_2+1}^{l\ l_1-1\ l_2}\right.$$
$$-\sqrt{(l_1-m_1)(l_1-m_1-1)(l_2+m_2)(l_2-m_2+1)}\hat{C}_{m\,m_1+1\,m_2-1}^{l\ l_1-1\ l_2} \qquad (B.16)$$
$$\left.-2\sqrt{(l_1+m_1)(l_1-m_1)}\,m_2\hat{C}_{m\,m_1\,m_2}^{l\ l_1-1\ l_2}\right).$$

The requirement of gauge invariance implies that the quantity in brackets is proportional to the Clebsch-Gordan coefficient $\hat{C}_{m\,m_1\,m_2}^{l\,l_1\,l_2}$, with the proportionality coefficient depending only on l_1, l_2, l. In other words, we must have:

$$\int \{\Theta_{m_1}^{l_1}, \Theta_{m_2}^{l_2}\}\overline{\Theta_m^l} = P(l_1, l_2, l)\hat{C}_{m\,m_1\,m_2}^{l\,l_1\,l_2}. \qquad (B.17)$$

The coefficient $P(l_1, l_2, l)$ is proportional to $\hat{C}_{00\ 0}^{ll_1-1\,l_2}$ and thus is non-zero only when $l_1+l_2+l = 2g-1, g \in \mathbb{Z}$. Since

$$\hat{C}_{m\,m_2\,m_1}^{l\,l_2\,l_1} = (-1)^{l-l_1-l_2}\hat{C}_{m\,m_1\,m_2}^{l\,l_1\,l_2}, \qquad (B.18)$$

for values of l_1, l_2, l summing up to an odd integer the Clebsch-Gordan coefficient changes sign under the exchange of l_1 with l_2. Because the Poisson bracket must be anti-symmetric, the coefficient $P(l_1, l_2, l)$ must be symmetric under the exchange of l_1 with l_2. Let us now determine this coefficient. It can be determined, for example, by choosing $l = m$ and using:

$$\hat{C}_{l\,j\,l-j}^{l\,l_1\,l_2} = (-1)^{l_1-j}\frac{(l_1+l_2-l)!}{(l_1+l_2+l+1)!\Delta(l_1,l_2,l)}\left[\frac{(l_1+j)!(l_2+l-j)!}{(l_1-j)!(l_2-l+j)!}\right]^{1/2}, \qquad (B.19)$$

where

$$\Delta(l_1, l_2, l) = \left[\frac{(l_1+l_2-l)!(l_1-l_2+l)!(l_2-l_1+l)!}{(l_1+l_2+l+1)!}\right]^{1/2}. \qquad (B.20)$$

After some algebraic manipulations this gives:

$$P(l_1, l_2, l) = (-1)^{\frac{l_1+l_2-l+1}{2}}(2g+2)\frac{g!\Delta(l_1,l_2,l)}{(g-l_1)!(g-l_2)!(g-l)!}. \qquad (B.21)$$

Here $2g+1 = l_1+l_2+l$ and we have used the formula (A.6) for $\hat{C}_{00\ 0}^{ll_1-1\,l_2}$. Expression (B.17) together with (B.21) is our final result for the Poisson bracket of two spherical harmonics. Let us also note that $P(l_1, l_2, l)$ can be written as a certain Clebsch-Gordan coefficient:

$$P(l_1, l_2, l) = -\sqrt{l_1(l_1+1)l(l+1)}\hat{C}_{1\,1\,0}^{ll_1l_2}. \qquad (B.22)$$

This equality is analogous to formula the (A.6).

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